

An algorithm to construct Gröbner bases for solving integration by parts relations

Alexander V. Smirnov*

Scientific Research Computing Center of Moscow State University

E-mail: asmirnov@rdm.ru

ABSTRACT: This paper is a detailed description of an algorithm based on a generalized Buchberger algorithm for constructing Gröbner-type bases associated with polynomials of shift operators. The algorithm is used to calculate Feynman integrals and has proved to be efficient in several complicated cases.

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Contents

1.	Introduction	1
2.	Preliminaries	2
3.	The s-reduction algorithm	4
4.	Constructing s-bases	6
5.	Example	8
6.	Conclusion	9

1. Introduction

In the framework of perturbation theory quantum-theoretical amplitudes are written as sums of Feynman integrals that are constructed according to Feynman rules. After a tensor reduction based on projectors is performed, each Feynman graph generates various scalar Feynman integrals with the same structure of the integrand and with various distributions of the powers of propagators (also called *indices*):

$$F(a_1, \dots, a_n) = \int \dots \int \frac{d^d k_1 \dots d^d k_h}{E_1^{a_1} \dots E_n^{a_n}}. \tag{1.1}$$

Here k_i , $i = 1, \dots, h$, are loop momenta and the denominators E_r are either quadratic or linear with respect to the loop momenta $p_i = k_i$, $i = 1, \dots, h$ or to the independent external momenta $p_{h+1} = q_1, \dots, p_{h+N} = q_N$ of the graph. Irreducible polynomials in the numerator can be represented as denominators raised to negative powers. Usual prescriptions $k^2 = k^2 + i0$, etc. are implied. The dimensional regularization [1] with $d = 4 - 2\varepsilon$ is assumed.

In today's perturbative calculations, when one needs to evaluate millions of Feynman integrals (1.1), a well-known strategy is to derive certain relations between Feynman integrals of a given family without calculating them, and then to apply the latter *recurrently*. As a rule one applies the so-called integration by parts (IBP) relations [2]

$$\int \dots \int d^d k_1 d^d k_2 \dots \frac{\partial}{\partial k_i} \left(p_j \frac{1}{E_1^{a_1} \dots E_n^{a_n}} \right) = 0. \tag{1.2}$$

After differentiating, the scalar products $k_i \cdot k_j$ and $k_i \cdot q_j$ are expressed linearly in terms of the factors E_i of the denominator, and one obtains the IBP relations in the following form:

$$\sum c_i F(a_1 + b_{i,1}, \dots, a_n + b_{i,n}) = 0, \tag{1.3}$$

where $b_{i,j}$ are integers, c_i are polynomials in a_j , d , masses m_i and kinematic invariants, and $F(a_1, \dots, a_n)$ are Feynman integrals (1.1) of a given family.

One tries to use IBP relations to express a dimensionally regularized integral of a given family as a linear combination of some ‘irreducible’ integrals also called *master* integrals. Recently there have been made several attempts to systematize this reduction procedure, and, in particular, the so-called Laporta’s algorithm [3, 4] (there exists its implementation available to all [5]), as well as Baikov’s method (see [6, 7] and chapter 6 of [8]), have been introduced. A different approach to this problem is connected with the use of Gröbner bases [9], which was first applied in [10] where IBP relations were reduced to differential equations. The direct use of non-commutative Gröbner bases in the algebra generated by shift operators was first suggested in [11, 12]. In our previous paper [13] (see also [14] for a brief review) we presented still another approach based on Gröbner bases. It is characterized, among other things, by the use of additional information on a given family of Feynman integrals, in particular the boundary conditions, i.e. the conditions of the following form:

$$F(a_1, a_2, \dots, a_n) = 0 \text{ when } a_{i_1} < 0, \dots, a_{i_k} < 0 \tag{1.4}$$

for some set of indices i_j (e.g., we always have $F(a_1, a_2, \dots, a_n) = 0$, if all a_i are non-positive). Since the time of our previous publication I have made this algorithm more efficient by the introduction of the so-called s-form. Moreover, the algorithm now works about ten times faster. Here the latest version of the algorithm is described in detail (the paper [13] was designed to give an introduction to the method, and a number of definitions in it were not formal).

Let us describe the basic idea of the algorithm. It consists of two parts: the construction of the so-called sector-bases, or s-bases, and the s-reduction. To use this method for calculating Feynman integrals of a given a family one has to construct the s-bases in the ideal generated by IBP relations with the use of the first part of the algorithm. After that the bases are used in the s-reduction part to reduce any given integral to master-integrals.

In section 2 I introduce some notions that are necessary to describe the algorithm. In section 3 I explain how the s-reduction works for a given set of bases (not necessarily the s-bases, however, only if the s-bases are used in the algorithm, one has a finite number of master integrals). In section 4 I describe the algorithm used to construct the s-bases, and in section 5 the example of the use of the algorithm is given.

2. Preliminaries

To describe the algorithm we have to introduce some notions. Let \mathcal{K} be the field of rational functions of physical variables m_i , $q_i \cdot q_j$, d , and \mathcal{A} be the algebra¹ over \mathcal{K} generated by elements Y_i , Y_i^- and A_i with the following relations:

$$\begin{aligned} Y_i Y_j &= Y_j Y_i, & A_i A_j &= A_j A_i, & Y_i A_j &= A_j Y_i + \delta_{i,j} Y_i, \\ Y_i^- Y_j^- &= Y_j^- Y_i^-, & Y_i^- Y_j &= Y_j Y_i^-, & Y_i^- A_j &= A_j Y_i^- - \delta_{i,j} Y_i, & Y_i^- Y_i &= 1 \end{aligned} \tag{2.1}$$

¹An algebra over a field is a vector space over this field and a ring at the same time.

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise. For convenience we will write $(Y_i^-)^k = Y_i^{-k}$. Let \mathcal{F} be the field of functions of n integer arguments a_1, a_2, \dots, a_n . The algebra \mathcal{A} acts on this field², where

$$\begin{aligned} (Y_i \cdot F)(a_1, a_2, \dots, a_n) &= F(a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_n), \\ (A_i \cdot F)(a_1, a_2, \dots, a_n) &= a_i F(a_1, a_2, \dots, a_n). \end{aligned} \tag{2.2}$$

The left-hand sides of relations (1.3) can be represented as elements of the algebra \mathcal{A} applied to F ; we will denote these elements by f_1, \dots, f_k . Now, for $F(a_1, \dots, a_n)$ defined by (1.1), we have

$$f_i \cdot F = 0 \text{ or } (f_i \cdot F)(a_1, \dots, a_n) = 0 \tag{2.3}$$

for all i . Let us generate a (left) ideal \mathcal{I} by the elements f_1, \dots, f_k ³. We will call \mathcal{I} the *ideal of the IBP relations*. Obviously,

$$f \cdot F = 0, \text{ or } (f \cdot F)(a_1, \dots, a_n) = 0 \text{ for any } f \in \mathcal{I}. \tag{2.4}$$

Let us consider the set \mathcal{D} with elements $\{c_1, c_2, \dots, c_n\}$ where all c_i are equal to 1 or -1 . The elements of this set will be called *directions*. For any direction $\nu = \{c_1, \dots, c_n\}$ we will consider a region $\sigma_\nu = \{(a_1, \dots, a_n) : (a_i - 1/2)c_i > 0\}$ and call it a *sector*. Obviously the union of all sectors contains all integer points in the n -dimensional vector space and the intersection of any two sectors is an empty set. For a sector σ_ν we will say that its direction is ν .

We will say that an element $X \in \mathcal{A}$ is written in the *proper* form if it is represented as

$$X = \sum r_j(A_1, \dots, A_n) \prod_i Y_i^{d_{i,j}}, \tag{2.5}$$

where r_j are polynomials (with coefficients in \mathcal{K}) and $d_{i,j}$ are integers. (So, all the operators A_i are placed on the left from the operators Y_i .) Obviously any element $X \in \mathcal{A}$ has a unique proper form. We will say that an element of \mathcal{A} is a *monomial* if its proper form has a single non-zero coefficient function r_j .

Let $\mathbb{N}^n = \{(b_1, \dots, b_n)\}$ where all b_i are integers and $b_i \geq 0$. This is a semi-group⁴ (with respect to $(b_1, \dots, b_n) + (b'_1, \dots, b'_n) = (b_1 + b'_1, \dots, b_n + b'_n)$). We will say that an ordering on \mathbb{N}^n (denoted with the symbol \succ) is *proper* if

- i) for any $a \in \mathbb{N}^n$ not equal to $(0, \dots, 0)$ one has $a \succ (0, \dots, 0)$
- ii) for any $a, b, c \in \mathbb{N}^n$ one has $a \succ b$ if and only if $a + c \succ b + c$.

Let us fix a direction $\nu = \{c_1, c_2, \dots, c_n\}$ We will say that the ν -*degree* of a monomial $r(A_1, \dots, A_n) \prod_i Y_i^{d_i}$ is $\{d_1 c_1, \dots, d_n c_n\}$ if all the products $d_i c_i$ are non-negative and undefined otherwise.

²(i) for any $a \in \mathcal{A}$ and $f \in \mathcal{F}$ we have an element $a \cdot f \in \mathcal{F}$; (ii) for any $a, b \in \mathcal{A}$ and $f, g \in \mathcal{F}$ we have $(a + b) \cdot (f + g) = a \cdot f + a \cdot g + b \cdot f + b \cdot g$; (iii) for any $a, b \in \mathcal{A}$ and $f \in \mathcal{F}$ we have $(ab) \cdot f = a \cdot (b \cdot f)$.

³A (left) ideal \mathcal{I} in an algebra \mathcal{A} is such a set that (i) for any $a \in \mathcal{A}$ and $b \in \mathcal{I}$ we have $ab \in \mathcal{I}$; (ii) for any $a, b \in \mathcal{I}$ we have $a + b \in \mathcal{I}$.

An ideal generated by a set of elements is the minimal possible ideal containing them. For a finite set of elements f_1, \dots, f_k it is the set of all linear combinations $a_1 f_1 + \dots + a_k f_k$ for $a_i \in \mathcal{A}$.

⁴A semi-group is a set with an associative binary operation.

Let us choose a proper ordering. Take an element $X \in \mathcal{A}$ and write it in its proper form. Consider the set of ν -degrees of all the monomials in the decomposition of X . If this set is empty (all the degrees are undefined) we will say that the ν -highest degree of X is undefined as well, otherwise we will say that ν -highest degree of X is the highest element of that set in terms of the fixed ordering.

We will say that a direction $\{c_1, \dots, c_n\}$ is lower than $\{c'_1, \dots, c'_n\}$ if $c_1 \leq c'_1, \dots, c_n \leq c'_n$ and they are not equal. The same is said about the corresponding sectors. We will say that a sector σ is *trivial* if all the integrals $F(a_1, \dots, a_n)$ are zero for $(a_1, \dots, a_n) \in \sigma$ due to boundary conditions (1.4). The same will be said about the direction of the sector.

3. The s-reduction algorithm

Suppose that for each non-trivial sector σ_ν we are given an ordering and a finite basis $\{X_{\nu,1}, \dots, X_{\nu,k_\nu}\} \subset \mathcal{I}$.

We are going to describe an algorithm called s-reduction.

Input: a linear combination of integrals $F(a_1, \dots, a_n)$.

Output: another linear combination of integrals containing the integrals that could not be reduced by these bases (an integral $F(a_1, \dots, a_n)$ is called *irreducible* for the given sets of orderings and bases if the ν -reduction of $F(a_1, \dots, a_n)$ returns $F(a_1, \dots, a_n)$, where ν is such a direction that $(a_1, \dots, a_n) \in \sigma_\nu$).

If one chooses properly the orderings and the elements X then there is a finite number of irreducible integrals, therefore the s-reduction will always output a combination of them but such a choice is a non-trivial procedure. The algorithm that might construct such elements will be described in the next section. But first we have to explain what s-reduction is.

We will illustrate our method with an example in section 5. More examples can be found at my webpage http://www.srcc.msu.ru/nivc/about/lab/lab4_2/index_eng.htm.

S-reduction.

1. $L = \text{Input}; M = 0$.
2. **While** $L \neq 0$
 3. Let S be the set of sectors that contain some point (a_1, \dots, a_n) where $F(a_1, \dots, a_n)$ has a non-zero coefficient in the decomposition of L .
 4. Let ν be a direction such that $\sigma_\nu \in S$ and there is no other sector $\sigma_{\nu'} \in S$ such that σ_ν is lower than $\sigma_{\nu'}$.
 5. Let $L = L_1 + L_2$ where L_1 contains those and only those $F(a_1, \dots, a_n)$ where $(a_1, \dots, a_n) \in \sigma_\nu$.
 6. $N = \nu$ -reduction of L_1 by $\{X_{\nu,1}, \dots, X_{\nu,k_\nu}\}$ (to be described below)
 7. Let $N = N_1 + N_2$ where N_1 contains those and only those $F(a_1, \dots, a_n)$ where $(a_1, \dots, a_n) \in \sigma_\nu$.
 8. $L = L_2 + N_2; M = M + N_1$.
9. Output= M

In a few words we are using the ν -reduction of X starting from higher sectors. Basically, it is a formalization of the standard method people use to solve the IBP relations “by hand”. Now we are going to describe what ν -reduction is.

Input: direction ν ; an element L that is a linear combination of integrals $F(a_1, \dots, a_n)$ such that all $(a_1, \dots, a_n) \in \sigma_\nu$; a finite basis $\{X_1, \dots, X_k\} \subset \mathcal{I}$.

Output: another linear combination of integrals $F(a_1, \dots, a_n)$ such that $(a_1, \dots, a_n) \in \sigma_{\nu'}$ where $\nu' = \nu$ or ν' is lower than ν .

ν -reduction.

1. Let $p = \{p_1, \dots, p_n\} = \{(c_1 + 1)/2, \dots, (c_n + 1)/2\}$, where $\nu = \{c_1, \dots, c_n\}$.
2. Set $Y = 0$.
3. Let $X \in \mathcal{A}$ be the element obtained by replacing $F(a_1, \dots, a_n)$ with $\prod_i Y_i^{a_i - p_i}$ in L . One has $L = (X \cdot F)(p)$.
4. **While** $X \neq 0$
 5. Let U be the highest term of X , $U = C \prod_i Y_i^{d_i c_i}$.
 6. For all possible products $T = (\prod_i Y_i^{x_i}) \cdot X_j$ such that the proper form of T has a non-zero coefficient at $\prod_i Y_i^{d_i c_i}$

Do (there is a finite number of possibilities)

 7. Replace all A_i in the proper form of T with p_i .
 8. If for some j such that $c_j = -1$ there is a term of T with Y_j in a positive degree then continue the cycle with the next element.
 9. If the coefficient C' of T at $\prod_i Y_i^{d_i c_i}$ is equal to zero then continue the cycle with the next element.
 10. Take $Z = X - (C/C')T$. Let Z_1 be the ν -sector part of Z (the sum of monomials in the decomposition of Z that have a defined degree) and $Z_2 = Z - Z_1$.
 11. If the ν -highest degree of Z_1 is lower than the ν -highest degree of X then replace X with Z_1 , Y with $Y + Z_2$ and go to the start of the While cycle (step 4).
 12. Replace X with $X - U$ and Y with $Y + U$
13. Return $(Y \cdot F)(p)$

The idea of the algorithm is to represent a given linear combination as an element X of \mathcal{A} being applied to F and the value being taken in the corner of the sector. Then one tries all possible transformation of the obtained element that lower the ν -degree of X (this is a generalization of the standard reduction procedure). The point is that through all the algorithm the value $((X + Y) \cdot F)(p)$ is constant. This is based on the fact that $(Z \cdot F)(p) = 0$ for any $Z \in \mathcal{I}$. Note that the elements X and Y in their proper forms do not depend on A_i (because of the replacement in step 7). Therefore in step 10 we have $C' \in \mathcal{K}$ and $C' \neq 0$, so the division is possible. The step 12 is the place where the terms that cannot be reduced are “moved” from X to Y . If the basis is chosen properly, those terms correspond to a finite number of master integrals.

4. Constructing s-bases

Let us fix a non-trivial direction $\nu = \{c_1, c_2, \dots, c_n\}$ and let $p = \{p_1, \dots, p_n\} = \{(c_1 + 1)/2, \dots, (c_n + 1)/2\}$. Our task is to construct an *s-basis* for this direction, i.e. such a basis $\{X_1, \dots, X_k\} \subset \mathcal{I}$ that the ν -reduction for this sector has only a finite number of irreducible integrals.

The relations (1.3) provide us a basis of \mathcal{I} but generally it is not an s-basis. Moreover, one has to choose an appropriate ordering.

We are going to describe an algorithm that takes the relations (1.3) as input and aims to construct an s-basis. The results of the algorithm greatly depend on the choice of the ordering and in complicated cases it is difficult to find one allowing to construct the needed basis. Now suppose we have fixed an ordering and an initial basis $\{X_1, \dots, X_k\} \subset \mathcal{I}$. Let us describe our algorithm.

First of all let us define the *s-form* of an element $X \in \mathcal{A}$ (note that this definition and some more definitions below depend on the choice of the direction ν , but it is fixed in this section). So, the *s-form* of an element $X \in \mathcal{A}$ is an element T of the form $(\prod_i Y_i^{x_i}) \cdot X$ satisfying the following properties:

- (i) The ν -highest degree of T is defined and for any integer (y_1, \dots, y_n) such that $y_1 c_1 \geq 0, \dots, y_n c_n \geq 0$ the ν -highest degree of $(\prod_i Y_i^{y_i}) \cdot T$ is equal to the ν -highest degree of T plus (y_1, \dots, y_n) ;
- (ii) The ν -highest coefficient of $(\prod_i Y_i^{y_i}) \cdot T$ does not vanish when $A_1 = p_1, \dots, A_n = p_n$ for all integer (y_1, \dots, y_n) such that $y_1 c_1 \geq 0, \dots, y_n c_n \geq 0$;
- (iii) For all j such that $c_j = -1$ the degrees of Y_i in the proper form of T are non-positive;
- (iv) The numbers $(c_1 x_1, \dots, c_n x_n)$ are minimal possible for all (x_1, \dots, x_n) satisfying the properties (i)-(iii).

Let us reformulate this definition less formally. The properties (i), (ii) mean that this element has “enough” terms whose degrees lie in the sector σ_ν . It is needed so that this element can be used for the ν -reduction. The property (iii) is required to control that the use of this element in the ν -reduction does not output any elements that lie in sectors higher than ν .

Through all the algorithm we will store a basis of \mathcal{I} consisting of elements in the s-form. Let us describe how the reduction of an element of \mathcal{A} modulo a basis $\{X_1, \dots, X_k\} \subset \mathcal{I}$ works.

Reduction.

1. $Y =$ s-form of Input.
2. If $Y = 0$ then Return 0.
3. For all j such that all the numbers $d_i - d'_i$ are positive, where (d_1, \dots, d_n) is the ν -highest degree of Y and (d'_1, \dots, d'_n) is the ν -highest degree of X_j

Do

4. Let $Z = C' Y - C \prod_i Y_i^{c_i(d_i - d'_i)} \cdot X_j$, where C is the ν -highest coefficient of Y and C' is the ν -highest coefficient of $(\prod_i Y_i^{c_i(d_i - d'_i)}) \cdot X_j$.
5. $Z =$ s-form of Z .

6. If the ν -highest degree of Z is lower than ν -highest degree of Y then replace Y with Z and Go to step 2.
8. Return Y .

It is easy to see that this reduction stops working after a finite number of steps (there can't be an infinite sequence of decreasing degrees). Basically, this procedure is close to the standard reduction procedure in Buchberger algorithm. The difference is the usage of s-forms and the fact that the elements Y_j can have both positive and negative degrees.

Now we can describe the main algorithm. As it has been said earlier, it starts from a basis of \mathcal{I} , moreover, all elements are taken in their s-forms. The goal of the algorithm is to construct another basis such that the ν -reduction for this sector has only a finite number of unreducible integrals. Therefore, after an element is added to the basis or a basis element is replaced we are performing a test to verify this condition (Completion Criteria). It consists of checking, whether for any j there is such m that for all integer $l \geq m$ the element $F(p_1, \dots, p_{j-1}, p_j + lc_j, p_{j+1}, \dots, p_n)$ can be reduced modulo this basis in the ν -reduction algorithm.

Here is the main algorithm:

1. **While** not Completion Criteria
 2. If there is an element in the basis that can be reduced by some other element, replace it with the result of the reduction and restart the cycle.
 3. Choose a pair of elements of the basis X' and X'' .
 4. Choose the smallest possible integers (d_1, \dots, d_n) such that $d_j \geq d'_j$ and $d_j \geq d''_j$, where (d'_1, \dots, d'_n) is the ν -highest degree of X' and (d''_1, \dots, d''_n) is the ν -highest degree of X'' .
 5. Evaluate $Z = C''' \prod_i Y_i^{c_i(d_i - d'_i)} \cdot X' - C'' \prod_i Y_i^{c_i(d_i - d''_i)} \cdot X''$, where C'' is the ν -highest coefficient of $(\prod_i Y_i^{c_i(d_i - d'_i)}) \cdot X'$ and C''' is the ν -highest coefficient of $(\prod_i Y_i^{c_i(d_i - d''_i)}) \cdot X''$, — the s-polynomial of X' and X'' .
 6. $Z =$ Reduction of Z (the previous algorithm).
 7. If $Z \neq 0$ then add Z to the basis.

When implementing the present algorithm it is natural to store the information about the pairs where the s-polynomials have been evaluated to avoid repeating the same calculations. Of course, the choice performed at line 3 might be different, and the algorithm effectiveness greatly depends on this choice. One more improvement of the algorithm is the use of the symmetries of the diagram. Instead of evaluating an s-polynomial one might take an element symmetric to some element of the basis and reduce it the same way. All this is realized by introducing a function on the set of pairs of elements of the basis and another one on the set of all possible symmetric element to the elements of the basis. So, at line 3 we are choosing such an element to evaluate, that the value of this *choice function* is minimal. Currently the algorithm uses the choice function intended to minimize the degree of the resulting element, and it already makes the algorithm effective. The work on finding the optimal choice functions is in progress.

5. Example

Let us illustrate this method with a simple example. Let us consider the propagator integrals with the masses m and 0 .

$$F(a_1, a_2) = \int \frac{d^d k}{(k^2 - m^2)^{a_1} [(q - k)^2]^{a_2}}. \quad (5.1)$$

The integrals are zero if $a_1 \leq 0$. The corresponding IBP relations generate the following elements:

$$\begin{aligned} f_1 &= d - 2A_1 - A_2 - 2m^2 A_1 Y_1 - m^2 A_2 Y_2 + q^2 A_2 Y_2 - A_2 Y_2 Y_1^{-1} \\ f_2 &= A_2 - A_1 - m^2 A_1 Y_1 - q^2 A_1 Y_1 - m^2 A_2 Y_2 + q^2 A_2 Y_2 - A_2 Y_2 Y_1^{-1} + A_1 Y_1 Y_2^{-1}. \end{aligned}$$

The sectors $\sigma_{\{-1,-1\}}$ and $\sigma_{\{-1,1\}}$ are trivial, thus we are going to construct two s-bases.

Let us start with the sector σ_ν for $\nu = \{1, 1\}$. We will use the lexicographical ordering. The ν -highest degrees of both f_1 and f_2 are equal to $(1, 0)$, the ν -highest coefficients are $-2m^2 A_1$ and $-(m^2 + q^2)A_1$ correspondingly, therefore the elements f_1 and f_2 satisfy the first three conditions in the definition of the s-form. It is easy to see that multiplying f_1 or f_2 by Y_1^{-1} or Y_2^{-1} will result in an element not satisfying these requirements, therefore f_1 and f_2 coincide with their s-forms.

Let us show that any element $F(1+t, 1)$ for $t > 0$ can be reduced modulo f_1 in the ν -reduction algorithm (this is one of the two requirements in the Completion Criteria). The element X on line 3 of the algorithm is Y_1^t and coincides with its highest term. Let us take $T = Y_1^{t-1} f_1$ on line 6. The ν -highest degree of this element is $(t, 0)$ — the same that of X , therefore we can make a reduction step.

We will skip the proof that the elements $F(1+t, 1)$ for $t > 0$ cannot be reduced modulo f_1 or f_2 in the ν -reduction algorithm (that means that the Completion Criteria does not hold for the initial bases). Instead of that we are going to make a step of the main algorithm and construct a new element f_3 that will work for those integrals.

The element Z on line 5 of the main algorithm is equal to

$$\begin{aligned} Z &= -dm^2 A_1 - dq^2 A_1 + 2q^2 A_1^2 + 3m^2 A_1 A_2 + q^2 A_1 A_2 + (2m^2 A_1^2 Y_1)/Y_2 - m^4 A_1 A_2 Y_2 \\ &\quad + 2m^2 q^2 A_1 A_2 Y_2 - q^4 A_1 A_2 Y_2 - (m^2 A_1 A_2 Y_2)/Y_1 + (q^2 A_1 A_2 Y_2)/Y_1. \end{aligned}$$

This element does not satisfy the first requirement in the definition of the s-form; the s-form of Z is equal to $Y_2 Z$.

$$\begin{aligned} Y_2 Z &= 2m^2 A_1 Y_1 + 3m^2 Y_2 - dm^2 Y_2 + q^2 Y_2 - dq^2 Y_2 + 2q^2 A_1 Y_2 + 3m^2 A_2 Y_2 + q^2 A_2 Y_2 \\ &\quad - m^4 Y_2^2 + 2m^2 q^2 Y_2^2 - q^4 Y_2^2 - m^4 A_2 Y_2^2 + 2m^2 q^2 A_2 Y_2^2 - q^4 A_2 Y_2^2 \\ &\quad - (m^2 Y_2^2)/Y_1 + (q^2 Y_2^2)/Y_1 - (m^2 A_2 Y_2^2)/Y_1 + (q^2 A_2 Y_2^2)/Y_1. \end{aligned}$$

The ν -highest degree of this element is $(1, 0)$, and after one reduction step we obtain

$$\begin{aligned} f_3 &= -m^2 + dm^2 + q^2 + dq^2 - 2q^2 A_1 - 2m^2 A_2 - 2q^2 A_2 (A_2 - 1)/Y_1 \\ &\quad + (2A_1 + A_2 - d - 1)/Y_2 + (m^2 - q^2)^2 A_2 Y_2 + (m^2 - q^2) A_2 Y_2 / Y_1. \end{aligned}$$

Following the same way as before we show that any element $F(1, 1 + t)$ for $t > 0$ can be reduced modulo f_3 in the ν -reduction algorithm. This means that the Completion Criteria holds and we have constructed an s-basis f_1, f_2, f_3 .

However, the element $F(1, 1)$ cannot be reduced in the same way for the highest coefficients of all three elements in the bases vanish when applied to this point. This means that $F(1, 1)$ is a master integral.

Let us turn to $\nu = \{1, -1\}$. The elements f_1 and f_2 again coincide with their s-forms. However, their ν -highest degrees are now $(1, 0)$ and $(1, 1)$ correspondingly. Thus we can reduce the element f_2 . As a result of the reduction procedure it is replaced with the element

$$f_4 = m^2 - dm^2 - q^2 - dq^2 + 2q^2A_1 + 2m^2A_2 + 2q^2A_2 + (1 + d - 2A_1 - A_2)/Y_2 - (m^2 - q^2)^2A_2Y_2 + (1 - A_2)/Y_1 + (q^2 - m^2)A_2Y_2/Y_1.$$

Following the same way as in the first sector we show that Completion Criteria holds for $\{f_1, f_4\}$ in this sector. The element $F(1, 0)$ also turns out to be a master integral.

Let us now illustrate how the s-reduction works and reduce $F(1, 2)$ to master integrals. We start with $\nu = \{1, 1\}$ and apply the ν -reduction to $F(1, 2)$. This corresponds to $X = Y_2$ on line 3 in the ν -reduction algorithm. On line 10 we obtain

$$Z = ((3 - d)(m^2 + q^2) + (d - 2)/Y_2)/(m^2 - q^2)^2.$$

Thus $Z_1 = (3 - d)(m^2 + q^2)/(m^2 - q^2)^2$ and $Z_2 = (d - 2)Y_2/(m^2 - q^2)^2$. Z_1 cannot be reduced any longer so that the algorithm returns

$$(Z \cdot F)(p) = (3 - d)(m^2 + q^2)F(1, 1)/(m^2 - q^2)^2 + (d - 2)F(1, 0)/(m^2 - q^2)^2.$$

The attempt to apply the ν -reduction for $\nu = \{1, -1\}$ turns out to be useless, for $F(1, 0)$ is a master integral, therefore the answer is

$$F(1, 2) = (3 - d)(m^2 + q^2)F(1, 1)/(m^2 - q^2)^2 + (d - 2)F(1, 0)/(m^2 - q^2)^2.$$

6. Conclusion

The algorithm described above is close to Buchberger algorithm, since it is based on calculating s-polynomials and reductions. However, the difference is significant. The introduction of the s-form leads to a situation when generally one can have two elements of the same degree that cannot be reduced one by another in this algorithm. Therefore, a standard proof that the algorithm has to stop at a certain point is of no use here. To prove that this algorithm stops for any given family of Feynman integrals still remains a problem that, hopefully, will be solved in the nearest future. On the other hand, the present algorithm appears to be much more efficient to solve reduction problems for Feynman integrals than more or less straightforward generalization of Buchberger algorithm (see, e.g., [11]). This was demonstrated in [13] where the reduction problem was solved for Feynman integrals relevant to the two- and three-loop static quark potential, with the number of indices $n = 7$. (The well-known two loop results [15] were reproduced.) New results obtained with

the help of this algorithm in cases with the number of indices $n = 9$ are to be published soon [16]. Preliminary analysis shows that the algorithm can be successfully applied to problems with the number of indices up to $n = 12$.

An implementation of the s-reduction part of the algorithm in Mathematica can be found at http://www.srcc.msu.ru/nivc/about/lab/lab4_2/index_eng.htm, together with a number of examples including those s-bases that have been constructed for the problems considered in [13].

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